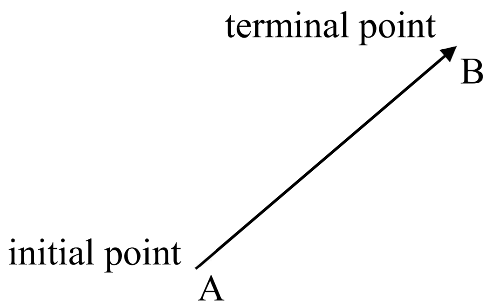
Chapter one

Vectors

Many physical quantities such as area, length, mass and temperature are completely described once the magnitude of the quantity is give, such quantities are called ***scalars***. other physical quantities called ***vectors*** are not completely determined until both a magnitude and a direction are specified such as force, velocity, and acceleration.

The vector represented by the directed line segment $\overrightarrow{AB}$ has initial point A and terminal point B



Vector in the plane:

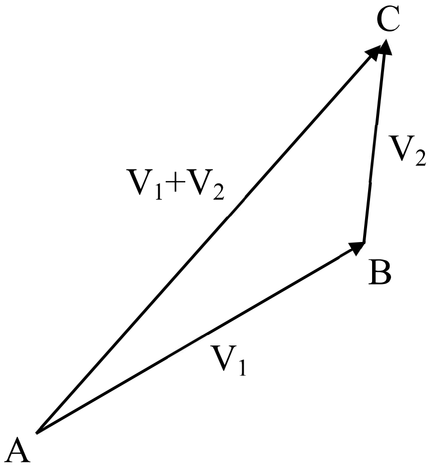
The vector between $P_1(x_1 , y_1)$ and $P_2(x_2 , y_2)$ is:

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j$$

Geometric addition (the parallelogram law):

Let $V_1 = \overrightarrow{AB}$, $V_2 = \overrightarrow{BC}$

Then $V_1 + V_2 = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

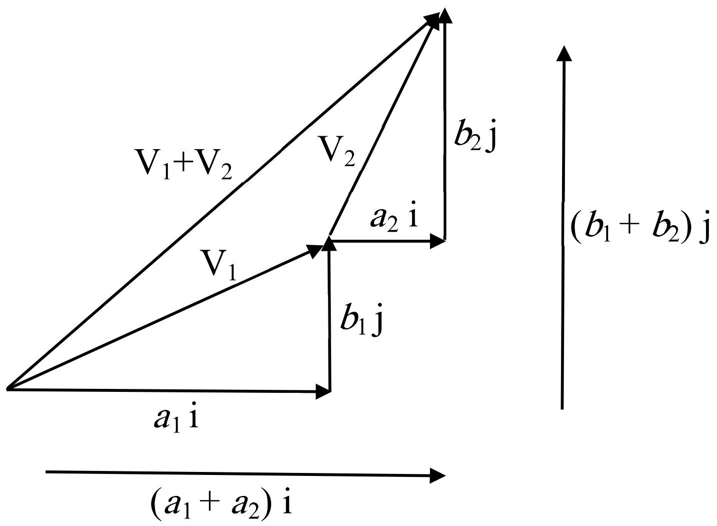


Algebra of vector:

***Algebraic addition*:**

Let $V_1 = a_1 i + b_1 j$

$V_2 = a_2 i + b_2 j$



Two vector may be added algebraically by adding their corresponding scalar components:

$$V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j$$

Example: If $V_1 = 2i - 5j$ and $V_2 = 4i + 2j$, find $V_1 + V_2$

solution:

$$V_1 + V_2 = (2i - 5j) + (4i + 2j)$$
$$= (2 + 4)i + (-5 + 2)j$$
$$= 6i - 3j$$

H.W: The vector $u = 4i + 3j$ and $v = 5i + 6j$, find $u + v$

Subtraction:

Let $V_1 = a_1 i + b_1 j$

$V_2 = a_2 i + b_2 j$

$$V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j$$

Example: If $V_1 = 7i + 3j$ and $V_2 = 2i - 6j$, find $V_1 - V_2$

solution:

$$\begin{aligned} V_1 - V_2 &= (7i + 3j) - (2i - 6j) \\ &= (7-2)i + (3-(-6))j \\ &= 5i + 9j \end{aligned}$$

H.W: The vector $u = 9i + 6j$ and $v = 5i + 2j$, find $u - v$

Length of the vector (magnitude):

The length of the vector is $V = ai + bj$ usually denoted by $|V|$, which may be read (the magnitude of V):

$$|V| = |ai + bj| = \sqrt{a^2 + b^2}$$

Example: find length of vector $V = 3i - 5j$

solution:

$$|V| = |3i - 5j| = \sqrt{(3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

H.W: The vector $u = 5i + 2j$, find the magnitude (length) of the vector u

***Multiplication by scalars*:**
The algebraic operation of multiplying a vector $V = ai + bj$ by a scalar C

$$\boxed{C(ai + bj) = (Ca)i + (Cb)j}$$

$$|CV| = |(Ca)i + (Cb)j| = \sqrt{(Ca)^2 + (Cb)^2}$$
$$= |C|\sqrt{a^2 + b^2} = |C||V|$$

***Example*:** Let C=2 and $V = -3i + 4j$ then:
***Solution*:**

$$|V| = |-3i + 4j| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

and

$$|2V| = |2(-3i + 4j)| = |-6i + 8j| = \sqrt{(-6)^2 + (8)^2} = \sqrt{36 + 64}$$
$$= \sqrt{100} = 10 = 2|V|$$

If C has been -2 instead of 2 , we would have found:

$$|-2V| = |-2(-3i + 4j)| = |6i - 8j| = \sqrt{(6)^2 + (-8)^2} = \sqrt{100} = 10$$
$$= |-2||V|$$

***Example*:** Let vectors $u = 3i + 2j$ and $v = -2i + 5j$ find: **1.** $3u$ **2.** $-2v$
***Solution*:**

1. $3u = 3(3i + 2j) = 3(3)i + 3(2)j$

$$= 9i + 6j$$

2. $-2v = -2(-2i + 5j) = -2(-2)i + (-2)(5)j$

$$= 4i - 10j$$

Zero vector:

The vector $0 = 0i + 0j$

Is called the zero vector. It is the only vector whose length is zero, as we can see from the fact that

$$|ai + bj| = \sqrt{a^2 + b^2} = 0$$
$$a = b = 0$$

Unit vector:

Any vector whose length is equal to the unit of length used along the coordinate axes is called a unit vector.

Direction:

$$\boxed{\text{Direction of the vector } V = \frac{V}{|V|}}$$

That is $\frac{V}{|V|}$ a unit vector in the direction of V , called the direction of the nonzero vector V.

Length of $\frac{V}{|V|} = \left|\frac{V}{|V|}\right| = \left|\frac{1}{|V|}V\right| = \frac{1}{|V|}|V| = 1$

The zero vector 0 has no defined direction

Example: find the direction of $A = 3i - 4j$

solution:

$$\text{Direction of } A = \frac{A}{|A|} = \frac{3i - 4j}{\sqrt{(3)^2 + (-4)^2}} = \frac{3i - 4j}{\sqrt{25}} = \frac{3}{5}i - \frac{4}{5}j$$

To check the length, we can calculate:

$$\left|\frac{3}{5}i - \frac{4}{5}j\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Properties of vector operations

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be vector and $\boldsymbol{a}$, $\boldsymbol{b}$ be scalars

1. $u + v = v + u$ **2.** $(u + v) + w = u + (v + w)$

3. $u + 0 = u$ **4.** $u + (-u) = 0$

5. $0u = 0$ **6.** $1u = u$

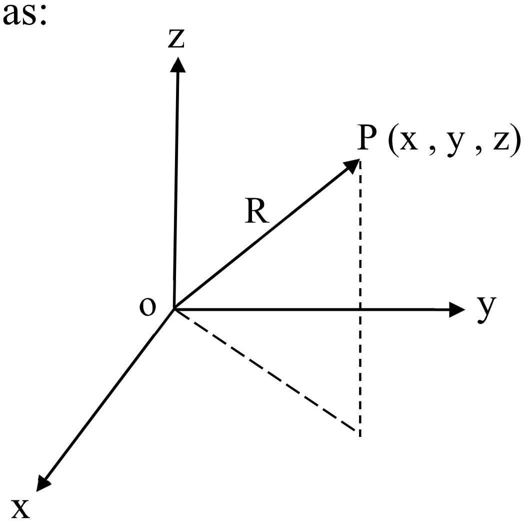
7. $a(bu) = (ab)u$ **8.** $a(u + v) = au + av$

9. $(a + b)u = au + bu$

***Vector in space*:**

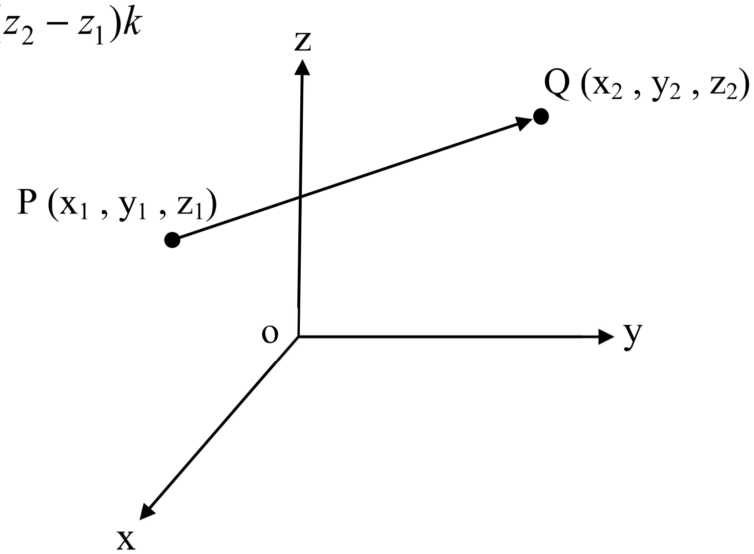
The vector R in space represented as:

$R = xi + yj + zk$



The vector between two points **P** and **Q**

$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$



***Algebra of vectors*:**

Let $V_1 = a_1 i + b_1 j + c_1 k$

$V_2 = a_2 i + b_2 j + c_2 k$

***Addition*:**

$$V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k$$

***Subtraction*:**

$$V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j + (c_1 - c_2)k$$

***The magnitude or length of the vector*:**

$V = ai + bj + ck$

$$|V| = \sqrt{a^2 + b^2 + c^2}$$

***Direction of vectors* V:** $\mathrm{V} = \dfrac{\mathrm{V}}{|\mathrm{V}|}$

***Example*:** find the length of vector $A = i - 2j + 3k$

***Solution*:**

$$|A| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{1 + 4 + 9} = \sqrt{14}$$

Example: find a unit vector *u* in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution:

$$\overrightarrow{P_1P_2} = (3-1)i + (2-0)j + (0-1)k$$
$$= 2i + 2j - k$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2)^2 + (2)^2 + (-1)^2}$$
$$= \sqrt{4+4+1} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2i+2j-k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Example: A force of 6N is applied in the direction of the vector $V = 2i + 2j - k$, express the force as a product of its magnitude and direction

Solution:

The force vector has magnitude 6 and direction $\frac{V}{|V|}$, so:

$$F = 6\frac{V}{|V|} = 6\frac{2i+2j-k}{\sqrt{(2)^2+(2)^2+(-1)^2}} = 6\frac{2i+2j-k}{\sqrt{9}}$$

$$= 6\frac{2i+2j-k}{3}$$

$$= 6\left(\frac{2}{3}i + \frac{2}{3}j + \frac{1}{3}k\right)$$

H.W:

1. Let the vector $u = -i + 3j + k$, find $\left|\frac{1}{2}u\right|$

2. find the length and direction of the vector $v = 4i + 3j + 2k$

3. find the length of the vector with initial point P(-3,4,1) and terminal point Q(-5,2,2)

Midpoint of a line segment:

The midpoint M of the line segment joining points $P_1(x_1 , y_1 , z_1)$ and $P_2(x_2 , y_2 , z_2)$ is the point:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example: the midpoint of the segment joining $P_1(3,-2,0)$ and $P_2(7,4,4)$ is:

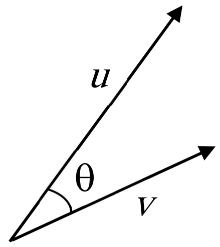
$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5,1,2)$$

***The dot product*:**

Dot product also called scalar products because the resulting products are numbers and not vectors. To calculate ***u.v*** from the component of ***u*** and ***v*** we let:

$$u = a_1 i + b_1 j + c_1 k$$

$$v = a_2 i + b_2 j + c_2 k$$



$$\boxed{u.v = |u||v|\cos\theta}$$

Where θ is the angle between ***u*** and ***v***

$$\boxed{\cos\theta = \frac{u.v}{|u||v|}}$$

$$\boxed{u.v = a_1 a_2 + b_1 b_2 + c_1 c_2}$$

Two vector ***u*** and ***v*** are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$ for such vectors, we have:

$u.v = 0 \Longrightarrow$ Because $\cos(\frac{\pi}{2}) = 0$

Properties of the dot product

if ***u*** , ***v***, and ***w*** are any vectors and ***c*** is a scalar:

1. $u.v = v.u$

2. $(cu).v = u.(cv) = c(u.v)$

3. $u.(v + w) = u.v + u.w$

4. $u.u = |u|^2$

5. $0.u = 0$

Example: find the angle θ between $A = i - 2j - 2k$ and $B = 6i + 3j + 2k$
Solution:

$$A.B = (1)(6) + (-2)(3) + (-2)(2)$$

$$= 6 - 6 - 4 = -4$$

$$|A| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|B| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-4}{21} \Longrightarrow \theta = \cos^{-1}\frac{-4}{21} = 100.79^o \approx 101^o$$

Example: find the angle between the vectors $u = 2i + j$, $v = i + 2j - k$
Solution:

$$u.v = (2)(1) + (1)(2) + (0)(-1)$$

$$= 2 + 2 - 0 = 4$$

$$|u| = \sqrt{(2)^2 + (1)^2 + (0)^2} = \sqrt{4+1+0} = \sqrt{5}$$

$$|v| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{u.v}{|u||v|} = \frac{4}{\sqrt{5}\sqrt{6}} = \frac{4}{\sqrt{30}}$$

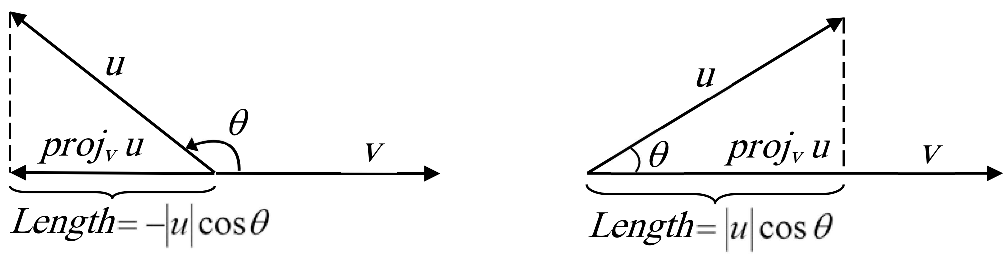
$$\theta = \cos{-1}\frac{4}{\sqrt{30}} = 43.09^o \approx 43^o$$

H.W: find the angle between the vectors $u = 2i - 2j + k$, $v = 3i + 4k$

***Vector projections and scalar components*:**

The vector we get by projecting a vector ***u*** onto the line through a vector ***v*** is called **(the vector projection of u onto v)**, sometimes denoted:

$$proj_v u \Longrightarrow \text{The vector projection of } \boldsymbol{u} \text{ onto } \boldsymbol{v}$$



The vector projection of ***u*** onto ***v*** is the vector:

$$proj_v u = \left(\frac{u.v}{|v|^2}\right) v$$

Where $|v|^2 = v.v$

The scalar component of ***u*** in the direction of ***v*** is the scalar:

$$|u|\cos\theta = \frac{u.v}{|v|} = u.\frac{v}{|v|}$$

***Example*:** find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v

***Solution*:** we find $proj_v u$ from equation:

$$proj_v u = \frac{u.v}{v.v} v = \frac{(6)(1) + (3)(-2) + (2)(-2)}{(1)(1) + (-2)(-2) + (-2)(-2)} (i - 2j - 2k)$$

$$= \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k)$$

$$= -\frac{4}{9} (i - 2j - 2k) = -\frac{4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

We find the scalar component of ***u*** in the direction of ***v*** from equation

$$|u| \cos\theta = u.\frac{v}{|v|}$$

$$|v| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$|u| \cos\theta = (6i + 3j + 2k).\frac{(i - 2j - 2k)}{3}$$

$$|u| \cos\theta = (6i + 3j + 2k).(\frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k)$$

$$= (6)(\frac{1}{3}) + (3)(-\frac{2}{3}) + (2)(-\frac{2}{3}) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$

Example: vectors $u = 5i + 12j$ and $v = \frac{3}{5}i + \frac{4}{5}k$, find:

1. the vector $proj_v u$

2. the scalar component of ***u*** in the direction of ***v***

Solution:

1.

$$proj_v u = \frac{u.v}{v.v} v$$

$$= \frac{(5)(\frac{3}{5}) + (12)(0) + (0)(\frac{4}{5})}{(\frac{3}{5})(\frac{3}{5}) + (0)(0) + (\frac{4}{5})(\frac{4}{5})} (\frac{3}{5}i + \frac{4}{5}k) = \frac{3}{\frac{9}{25} + \frac{16}{25}} (\frac{3}{5}i + \frac{4}{5}k)$$

$$= 3(\frac{3}{5}i + \frac{4}{5}k) = \frac{9}{5}i + \frac{12}{5}k$$

2.

$$|u|\cos\theta = u.\frac{v}{|v|}$$

$$|v| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

$$|u|\cos\theta = (5i + 12j).(\frac{3}{5}i + \frac{4}{5}k)$$

$$= (5)(\frac{3}{5}) + (12)(0) + (0)(\frac{4}{5})$$

$$= 3$$

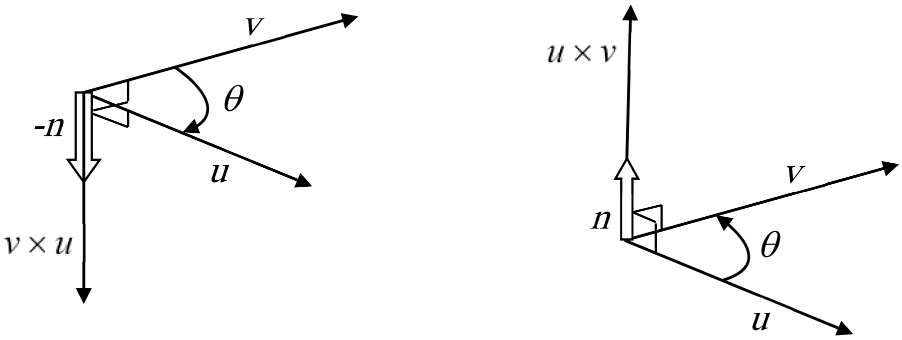
H.W: vectors $u = -2i + 4j - \sqrt{5}k$ and $v = 2i - 4j + \sqrt{5}k$, find:

1. the vector $proj_v u$

2. the scalar component of ***u*** in the direction of ***v***

Cross product:

Two vector ***u*** and ***v*** in space if ***u*** and ***v*** are not parallel, they determine a plane, we select a unit vector ***n*** perpendicular to the plane by the right-hand rule. This means that we choose ***n*** to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from ***u*** to ***v***



Then the cross product $u \times v$ is the vector defined as follows:

$$u \times v = (|u||v|\sin\theta)n$$

Properties of the cross product

if ***u*** **,** ***v*****,** and ***w*** are any vectors and ***r*** **,** ***s*** is are scalars, then:

1. $(ru)\times(sv) = (rs)(u \times v)$

2. $u \times (v + w) = u \times v + u \times w$

3. $v \times u = -(u \times v)$

4. $(v + w)\times u = v \times u + w \times u$

5. $0 \times u = 0$

6. $u \times (v \times w) = (u.w)v - (u.v)w$

When we apply the definition to calculate the pair wise cross products of ***i*** , ***j***, ***k*** we fined:

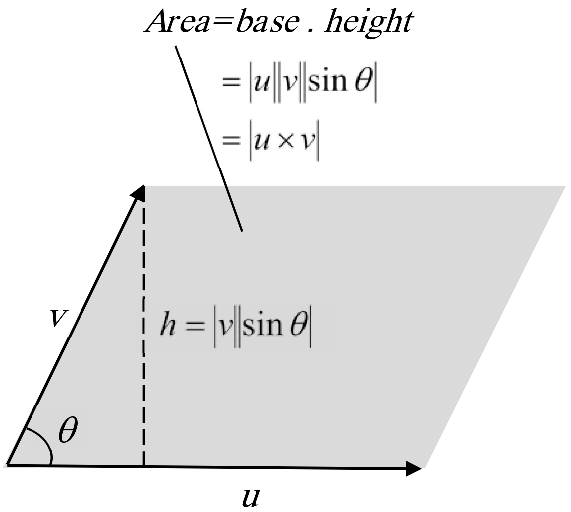
$$i \times j = -(j \times i) = k$$
$$j \times k = -(k \times j) = i$$
$$k \times i = -(i \times k) = j$$
$$i \times i = j \times j = k \times k = 0$$

$|u \times v|$ ***Is the area of a parallelogram***

Because n is a unit vector, the magnitude of $u \times v$ is:

$$|u \times v| = |u||v||\sin\theta||n| = |u||v|\sin\theta$$

This is the area of the parallelogram determined by ***u*** and ***v***, $|u|$ being the base of the parallelogram and $|v||\sin\theta|$ the height.



Example: find $u \times v$ and $v \times u$ if $u = 2i + j + k$ and $v = -4i + 3j + k$

Solution:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= ((1)(1) - (1)(3))i - ((2)(1) - (1)(-4))j + ((2)(3) - (1)(-4))k$$

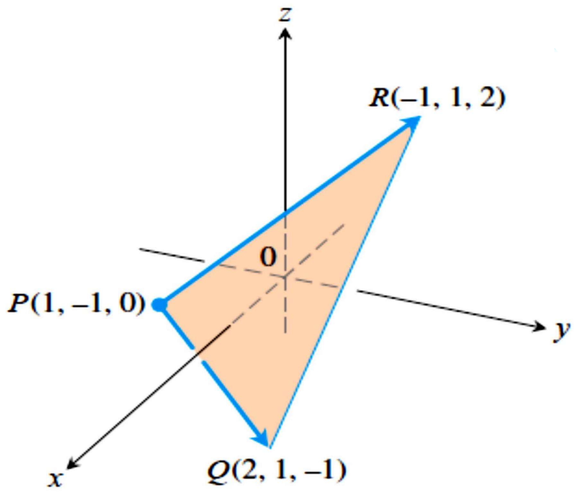
$$= (1-3)i - (2+4)j + (6+4)k$$

$$= -2i - 6j + 10k$$

$$v \times u = -(u \times v) = 2i + 6j - 10k$$

Example: find the area of the triangle with vertices p(1,-1,0) , Q(2,1,-1) and R(-1,1,2)

Solution:



The area of the triangle is

$(\frac{1}{2})\ |\overrightarrow{PQ} \times \overrightarrow{PR}|$. in term of components

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= ((2)(2) - (-1)(2))i - ((1)(2) - (-1)(-2))j + ((1)(2) - (2)(-2))k$$

$$= (4+2)i - (2-2)j + (2+4)k$$

$$= 6i + 6k$$

Hence, the triangle's area is:

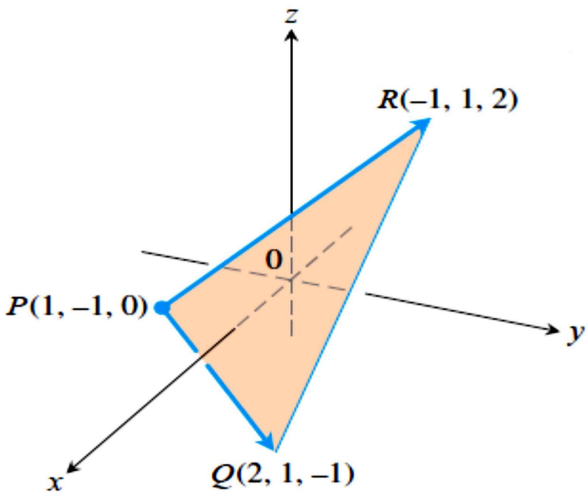
$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6k| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$$

The triangle's area is half of this $= 3\sqrt{2}$

Example: find a unit vector perpendicular to the plane of p(1,-1,0) , Q(2,1,-1) and R(-1,1,2)

Solution:

Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the Plane, its direction *n* is a unit vector Perpendicular to the plane



$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= 6i + 6k$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6k| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$$

$$n = \frac{6i + 6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k$$

H.W: Triangle with points $P(1,-1,2)$, $Q(2,0,-1)$ and $R(0,2,1)$, find:

1. Area of the triangle.

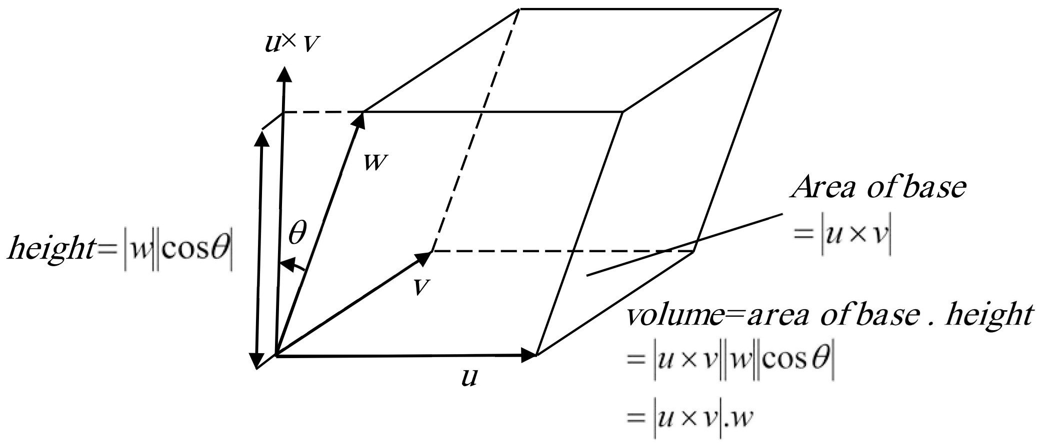
2. a unit vector perpendicular to the plane *PQR*.

Triple scalar or box product:

The product $(u\times v).w$ is called **the triple scalar product** of ***u***, ***v*** and ***w***.
As you can see from the formula:

$$|(u\times v).w| = |u\times v||w||\cos\theta|$$

The absolute value of this product is the volume of the parallelepiped (parallelogram – side box) determined by ***u***, ***v*** and ***w***. the number $|u\times v|$ is the area of the base parallelogram. The number $|w||\cos\theta|$ is the parallelepiped's height. Because this geometry, $(u\times v).w$ is called **the box product** of ***u***,***v*** and ***w***.



To calculate the triple scalar product:

$$Let\quad u = a_1 i + b_1 j + c_1 k$$
$$v = a_2 i + b_2 j + c_2 k$$
$$w = a_3 i + b_3 j + c_3 k$$

Then

$$(u\times v).w = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example: Find volume of box (parallelepiped) determined by

$$u = i + 2j - k \ , \ v = -2i + 3k \ \text{and} \ w = 7j - 4k$$

Solution:

$$(u \times v).w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix}(1) - \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix}(2) + \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}(-1)$$

$$= [(0)(-4) - (3)(7)](1) - [(-2)(-4) - (3)(0)](2) + [(-2)(7) - (0)(0)](-1)$$

$$= (0 - 21)(1) - (8 - 0)(2) + (-14 - 0)(-1)$$

$$= (-21)(1) - (8)(2) + (-14)(-1)$$

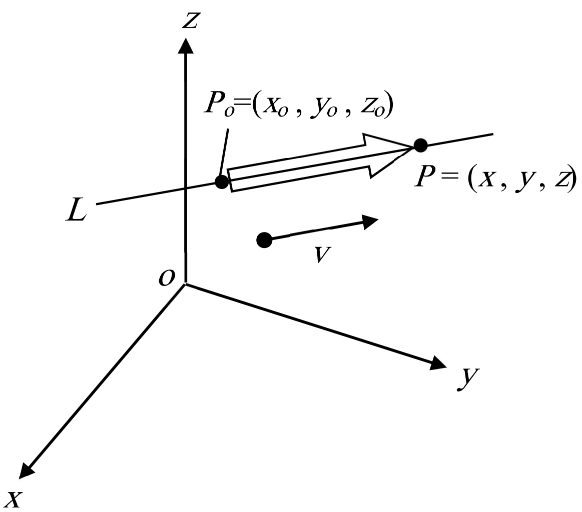
$$= -21 - 16 + 14 = \boxed{-23}$$

The volume is $|(u \times v).w| = 23$ units cubed

H.W: Find volume of box (parallelepiped) determined by $u = i + j - 2k$, $v = -i - k$ and $w = 2i + 4j - 2k$

Equation lines in space:

Suppose that ***L*** is a line in space through a point $P_o(x_o , y_o , z_o)$ Parallel to a vector $v = ai + bj + ck$,then ***L*** is the set of all points $p\ (x , y , z)$ for which $\overrightarrow{P_oP}$ is parallel to ***v***. thus $\overrightarrow{P_oP} = tv$ for some scalar parameter ***t***. the value of ***t*** depend on the location of the point ***P*** along the line. the expanded from of the equation:



$\overrightarrow{P_oP} = tv$

$(x - x_o)i + (y - y_o)j + (z - z_o)k = t(ai + bj + ck)$

$x - x_o = ta$

$y - y_o = tb$

$z - z_o = tc$

From equation above:

The parametric equation for the line through $P_o(x_o, y_o, z_o)$ parallel to $v = ai + bj + ck$ **:**

$$x = x_o + ta$$
$$y = y_o + tb$$
$$z = z_o + tc$$

***Example*:** Find parametric equations for the line through the point $(-2,0,4)$ parallel to the vector $v = 2i + 4j - 2k$

***Solution*:**

With $P_o(x_o, y_o, z_o) = (-2,0,4)$

$x_o = -2 \quad , \; y_o = 0 \quad , \; z_o = 4$

and $v = ai + bj + ck = 2i + 4j - 2k$

$a = 2 \quad , \; b = 4 \quad , \; c = -2$

$\therefore$

$x = x_o + at \Longrightarrow x = -2 + 2t$

$y = y_o + bt \Longrightarrow y = 4t$

$z = z_o + ct \Longrightarrow z = 4 - 2t$

Example: find parametric equation for the line through the points $P(-3,2,-3)$ and $Q(1,-1,4)$

Solution: the vector

$$\overrightarrow{PQ} = (1-(-3))i + (-1-2)j + (4-(-3))k$$

$$\overrightarrow{PQ} = 4i - 3j + 7k$$

$$\therefore \quad a = 4 \quad , \quad b = -3 \quad , c = 7$$

The vector is parallel to the line with chose ***P*** as the (**base point**):

$$P_o(x_o\,, y_o\,, z_o) \Longrightarrow P(-3,2,-3)$$

$$\therefore \quad x_o = -3 \quad , \quad y_o = 2 \quad , z_o = -3$$

$$\therefore \quad x = x_o + at \Longrightarrow x = -3 + 4t$$

$$y = y_o + bt \Longrightarrow y = 2 - 3t$$

$$z = z_o + ct \Longrightarrow z = -3 + 7t$$

We could have ***Q*** as the base point

***H.W*:**

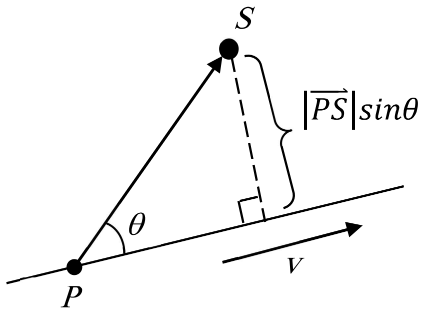
1. Find parametric equation for the line through the point $P(3,-4,-1)$ parallel to the vector $v = i + j + k$

2. Find parametric equation for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$

The distance from a point to a line in space:

To find the distance from a point ***S*** to a line that passes through a point ***P*** parallel to a vector ***v*** :

$$d = \frac{|\overrightarrow{PS} \times v|}{|v|}$$



Example: Find the distance from the point $S(1,1,5)$ to the line

$$L:\ x = 1 + t \quad , \ y = 3 - t \quad , \ z = 2t$$

Solution: from the equations for L

$$L:\ x = 1 + t \quad , \ y = 3 - t \quad , \ z = 2t$$

$$x = x_o + at \ , \ y_o + bt \ , \ z = z_o + ct$$

$$\therefore \quad x_o = 1 \ , \ y_o = 3 \ , z_o = 0 \Longrightarrow \boxed{P(1,3,0)}$$

$$a = 1 \ , \ b = -1 \ , \ c = 2$$

$$and \ \ v = ai + bj + ck \Longrightarrow \boxed{v = i - j + 2k}$$

$$\overrightarrow{PS} = (1-1)i + (1-3)j + (5-0)k$$
$$\overrightarrow{PS} = -2j + 5k$$

$$and \quad \overrightarrow{PS} \times v = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix} i - \begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} j + \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} k$$

$$= [(-2)(2) - (5)(-1)]i - [(0)(2) - (5)(1)]j + [(0)(-1) - (-2)(1)]k$$
$$= (-4 + 5)i - (0 - 5)j + (0 + 2)k$$
$$= i + 5j + 2k$$

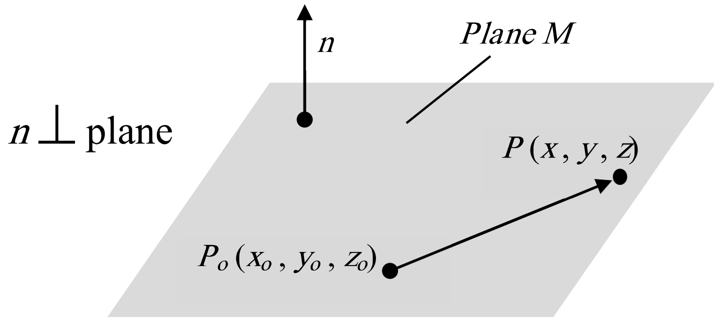
$$d = \frac{|\overrightarrow{PS} \times v|}{|v|} = \frac{\sqrt{(1)^2 + (5)^2 + (2)^2}}{\sqrt{(1)^2 + (-1)^2 + (2)^2}} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

H.W: Find the distance from the point $S(0,0,12)$ to the line

$$L:\ x = 4t \quad , \ y = -2t \quad , \ z = 2t$$

Equation for plane in space:

Suppose that plane ***M*** passes through a point $P_o(x_o , y_o , z_o)$ and is normal to the nonzero vector $n = Ai + Bj + Ck$, then ***M*** is the set of all points $P(x,y,z)$ for which $\overrightarrow{P_oP}$ is orthogonal to ***n***. Thus the dot product



$n.\overrightarrow{P_oP} = 0$

This equation is equivalent to:

$$(Ai + Bj + Ck).[(x - x_o)i + (y - y_o)j + (z - z_o)k] = 0$$

$$\boxed{A(x - x_o) + B(y - y_o) + C(z - z_o) = 0} \Longrightarrow \textit{Component equation}$$

This becomes:

$$Ax + By + Cz = Ax_o + By_o + Cz_o$$

When rearranged or more simply:

$$\boxed{Ax + By + Cz = D} \Longrightarrow \textit{Component equation simplified}$$

Where $D = Ax_o + By_o + Cz_o$

Example: Find an equation for the plane through $P_o(-3,0,7)$ perpendicular to $n = 5i + 2j - k$

Solution:

$A = 5 \quad , \quad B = 2 \quad , C = -1$

$x_o = -3 \quad , \; y_o = 0 \quad , \; z_o = 7$

$Ax + By + Cz = D$

Where $D = Ax_o + By_o + Cz_o$

$= (5)(-3) + (2)(0) + (-1)(7) = -15 + 0 - 7 = -22$

$\therefore \quad 5x + 2y - z = -22$

Example: Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$

Solution: we find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane:

$$\overrightarrow{AB} = 2i - k$$

$$\overrightarrow{AC} = 3j - k$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} k$$

$$= \left[(0)(-1) - (-1)(3)\right] i - \left[(2)(-1) - (-1)(0)\right] j + \left[(2)(3) - (0)(0)\right] k$$

$$= (0+3)i - (-2-0)j + (6-0)k$$

$$= 3i + 2j + 6k$$

$$\therefore \quad A = 3 \quad , \quad B = 2 \quad , C = 6$$

$$P_o = A(0,0,1) \Longrightarrow x_o = 0 \quad , y_o = 0 \quad , z_o = 1$$

$$\boxed{Ax + By + Cz = D}$$

Where $D = Ax_o + By_o + Cz_o$

$$= (3)(0) + (2)(0) + (6)(1)$$

$$= 0 + 0 + 6 = 6$$

$$\therefore \quad 3x + 2y + 6z = 6$$

As an equation for the plane

Example: Find the point where the line

$$x = \frac{8}{3} + 2t \quad , \; y = -2t \quad , \; z = 1 + t \quad \text{intersects the plane}$$

$$3x + 2y + 6z = 6$$

Solution:

The point $(\frac{8}{3} + 2t \, , \, -2t \; , \; 1 + t)$ lies in the plane if its coordinates satisfy the equation of the plane, that is if:

$$3(\frac{8}{3} + 2t) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t + 14 = 6$$

$$8t = 6 - 14$$

$$8t = -8 \quad \Longrightarrow \quad \boxed{t = -1}$$

The point intersection is:

$$(x, y, z)_{t=-1} = (\frac{8}{3} - 2, 2, 1 - 1) = (\frac{2}{3}, 2, 0)$$

H.W:

1. Find an equation for the plane through $P_o(0,2,-1)$ perpendicular to $n = 3i - 2j - k$

2. Find an equation for the plane through $P(1,1,-1)$, $Q(2,0,2)$ and $S(0,-2,1)$

The distance from a point to a plane:

If ***P*** is a point on a plane with normal ***n***, then the distance from any point ***S*** to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto ***n***. that is, the distance from ***S*** to the plane:

$$\boxed{d = \left| \overrightarrow{PS} . \frac{n}{|n|} \right|}$$

Where $n = Ai + Bj + Ck$ is the normal to the plane

Example: Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z=6$

Solution: the distance from ***S*** to the plane is:

$$d = \left|\overrightarrow{PS}.\frac{n}{|n|}\right|$$

From equation of plane $Ax+By+Cz=D$

$$3x+2y+6z=6$$

$$\therefore \quad , \quad B=2 \quad , C=6 \; A=3$$

$$n = Ai+Bj+Ck \Longrightarrow \therefore \boxed{n = 3i+2j+6k}$$

The points on the plane easiest to find from the plane's equation are the intercepts. If take *P* to the *y*-intercept (0,3,0), then:

$$\overrightarrow{PS} = (1-0)i+(1-3)j+(3-0)k \Longrightarrow \boxed{\overrightarrow{PS} = i-2j+3k}$$

$$|n| = \sqrt{(3)^2+(2)^2+(6)^2} = \sqrt{9+4+36} = \sqrt{49} = \boxed{7}$$

$$\therefore \quad d = \left|(i-2j+3k).\frac{(3i+2j+6k)}{7}\right|$$

$$= \left|(i-2j+3k).(\frac{3}{7}i+\frac{2}{7}j+\frac{6}{7}k)\right|$$

$$= \left|(1)(\frac{3}{7})+(-2)(\frac{2}{7})+(3)(\frac{6}{7})\right|$$

$$= \left|\frac{3}{7}-\frac{4}{7}+\frac{18}{7}\right| = \boxed{\frac{17}{7}}$$

***H.W*:**

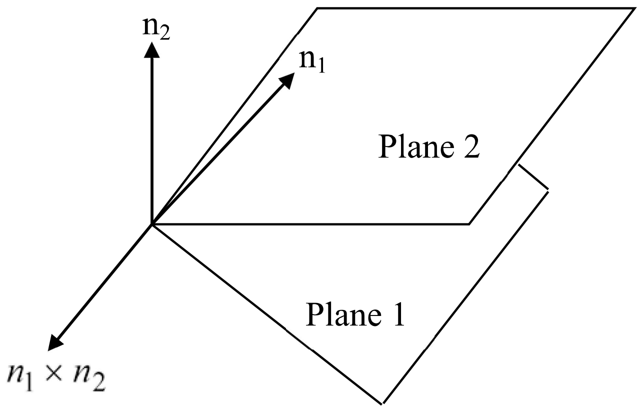
1. Find the distance from the point $S(2,-3,4)$ to the plane $x+2y+2z=13$

2. Find the distance from the point $S(0,-1,0)$ to the plane $2x+y+2z=13$

Angle between planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors.

$$\theta=\cos^{-1}\left(\frac{n_1.n_2}{|n_1||n_2|}\right)$$



Example: Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution: from equation of plane $Ax+By+Cz=D$

$$n = Ai + Bj + Ck$$

For plane 1: $A=3$, $B=-6$, $C=-2$

$$\therefore \quad \boxed{n_1 = 3i - 6j - 2k}$$

For plane 2: $A=2$, $B=1$, $C=-2$

$$\therefore \quad \boxed{n_2 = 2i + j - 2k}$$

are normal to the planes. The angle between them is:

$$\theta = \cos^{-1}\left(\frac{n_1.n_2}{|n_1||n_2|}\right)$$

$$n_1.n_2 = (3i-6j-2k).(2i+j-2k)$$

$$n_1.n_2 = (3)(2)+(-6)(1)+(-2)(-2) = 6-6+4 = \boxed{4}$$

$$|n_1| = \sqrt{(3)^2+(-6)^2+(-2)^2} = \sqrt{9+36+4} = \sqrt{49} = \boxed{7}$$

$$|n_2| = \sqrt{(2)^2+(1)^2+(-2)^2} = \sqrt{4+1+4} = \sqrt{9} = \boxed{3}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^o$$

H.W: find the angle between the planes $x+y=1$ and $2x+y-2z=2$